

## Considerations on travelling wave solutions concerning travelling bands of chemotactic microorganisms

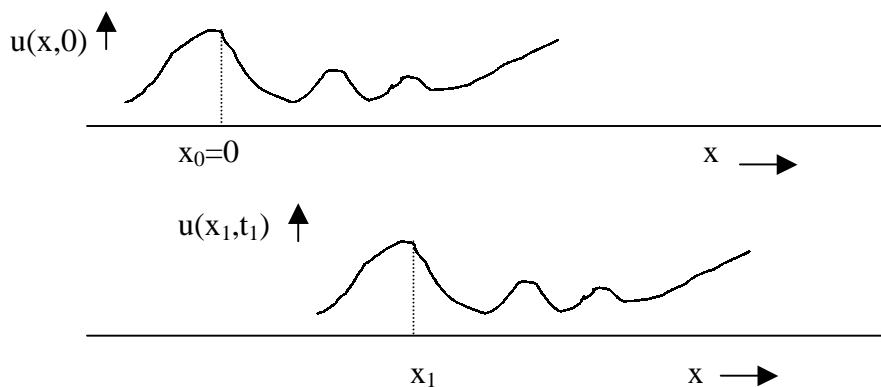
### Introduction

It has been already established that many biological phenomena exhibit, as their most apparent feature, a coherent pattern or waveform that moves in space. Such practical examples can be the depolarisation waves propagating along nerve axons, coherent swarms of motile micro-organisms advancing steadily through their environment toward a fresh supply of diffusing nutrient which they consume and seek chemotactically, chemical concentration waves carried by fluid buffer flow and diffusion through a separation column, propagation by random motility of logistically reproducing organisms through a one dimensional universe. All these biological examples have been already more or less transposed into theoretical mathematical formulas. In this particular case the mathematical representation of the propagating waves will be investigated. The present essay will treat the existence of the travelling wave solutions in the case of the chemotactic microorganisms, namely coherent moving bands or swarms that move towards the points where the food is concentrated.

### Maths intro

Before stating/defining any mathematical model we are going to see how we can mathematically define and describe propagating patterns in general. A rather trivial way of defining the travelling waves is the following: By the concept of travelling waves we understand a particular class of solutions to differential equation systems, characterised by distributions that move over space while maintaining a characteristic shape or profile. For a more insightful perspective, we are going to perform the following steps. Let us suppose that  $u(x, t)$  represents the variation with the position  $x$  and the time  $t$  of something involved in a propagating wave phenomenon.

(1)



In the figure above two different positions of an arbitrary waveform propagating along the x-axis at a constant speed is described.

As seen in the figures above, we observe two snapshots, one at the instant  $t=0$  and the other at a later time  $t=t_1$ , where, definitely,  $t_1>0$ . We suppose that the curve illustrated at  $t=0$  moves steadily down the  $x$ -axis, therefore in the figures, the curve illustrated at  $t=t_1$  is geometrically congruent with the curve at  $t=0$ . In the same figures,  $x_0=0$  and  $x_1$  locates the positions of maximum concentration at times  $t=0$  and respectively  $t=t_1$ . This way, we can immediately have the speed of propagation of this wave. Let us denote it by  $c$ :

$$c = \frac{x_1 - x_0}{t_1 - 0} = \frac{x_1}{t_1}$$

If we view this waveform from a co-ordinate system that moves at speed  $c$  (so the speed calculated above), then, in the moving coordinate system, the wave shape will not change with time. Let  $z$  measure distance, parallel to the  $x$ -axis, from the centre of this moving co-ordinate system. If we denote by  $U(z)$  the shape of the waveform as seen from this system,  $U$  will not change with time. We can find a relation between  $U(z)$  and  $u(x, t)$  and this relation is the following:

(3)

$$U(z) = u(x, 0)$$

This is because at time  $t=0$ ,  $x=0$  and  $z=0$  locate the same position. Further on, we must have

(4)

$$U(0) = u(x_1, t_1)$$

and by generalisation,

(5)

$$U(z) = u(x_1 + z, t_1)$$

Because at time  $t=t_1$ ,  $z=0$  and  $x=x_1$  locate precisely the same position. Now, having the speed of the wave already calculated, we have

(6)

$$U(z) = u(ct_1 + z, t_1)$$

We can write it in a more general form (for every time  $t$ ):

(7)

$$U(z) = u(ct + z, t)$$

If now we identify  $x=ct+z$ , this being implicitly  $z=x-ct$ , we arrive at

(8)

$$u(x, t) = U(z) = U(x - ct)$$

Now, this last relationship is exactly what we sought in the beginning, and proves that  $u(x, t)$  represents a fixed waveform propagating along the  $x$  axis at constant speed  $c$ , if and only if, for some function  $U(z)$ , the last relation written above (8) holds. We call  $z=x-ct$  a wave variable.

### Biological intro

A brief intro to the biological side of this subject could be appropriate before introducing the mathematical description. In general, by chemotaxis we understand a phenomenon that occurs when an organism moves preferentially toward a relatively

high concentration of some chemical (and then we talk about positive chemotaxis) or away from such a concentration (implicitly, negative chemotaxis). Many unicellular organisms present elaborate patterns of locomotion that may include diverse ways, some of them being ciliary beating (that is, synchronous motion of hair-like appendages on the cell surface), helical swimming, crawling on surfaces, tumbling in three dimensions or pseudopodial extension (that is, protrusion of part of the cell and streaming of the cellular contents). Referring to the appearance of these motions, in the absence of overriding external cues, the motion may appear saltatory (jerky) or random; eventually, these motion features are determined by events on subcellular levels. All these terms can be discussed in more detail, but this is not the subject of this paper. Given the information above we are trying to find travelling wave solutions in the motion of these microorganisms. As an observation, amoebae and bacteria are prime model systems for investigating chemotaxis at the population and molecular levels.

## **Bacterial chemotaxis**

Bacterial chemotaxis is a phenomenon that is currently under investigation. An intuitive idea about the experiments that have been done in order to emphasise the chemotaxis can be summarised in a few lines. A one-dimensional universe (a long capillary tube) is filled with a fluid medium in which a nutrient substance,  $s$ , is dissolved. Bacteria are inoculated into one end of the tube. They are observed to consume the nutrient in their neighbourhood and to form a band that moves steadily up the tube. Now, what can be the reason for a need of a mathematical model in this case? After all, at first sight, if the bacteria consume the local nutrient supply, and can sense abundant food nearby down the tube, and have this ability to move toward it, then what else can happen? However, by performing a mathematical analysis of the phenomenon, it can be observed that only certain special chemotactic response algorithms will lead to band propagation maintaining coherence over “long” (compared to band width) distances. Therefore, the mathematical quest will mainly consist on cataloguing all the macroscopic (or phenomenological) behaviour repertoires of chemotactic microorganisms that lead to propagating bands.

### **The constitutive equations for chemotaxis**

Keller and Segel were among the first to describe a continuum equation underlying the phenomenon of chemotaxis in the case of the microorganisms. The basis of their model was constituted by the ideas of attraction and repulsion. If we denote with  $b(x, t)$  the population density of bacteria and with  $s(x, t)$  the nutrient density (actually derived from the substrate concentration), then we can state a hypothesis about the constitutive relation for the bacterial flux density:

(9)

$$J_b = -\mu \frac{\partial b}{\partial x} + b\chi \frac{\partial s}{\partial x}$$

The first term of the relation represents the random (diffusive) component of the flux:

(10)

$$J_{random} = -\mu \frac{\partial b}{\partial x} + b\chi \frac{\partial s}{\partial x}$$

The second component is the chemotactic component of the flux;

(11)

$$J_{chemotactic} = -\mu \frac{\partial b}{\partial x} + b\chi \frac{\partial s}{\partial x}$$

Now, the model is to be restricted at the cases when the bacteria neither die, nor reproduce during the experiment (reproduction can be prevented chemically).

We can now agree on a balance law for the bacterial density  $b(x, t)$ .

(12)

$$\frac{\partial b}{\partial t} = -\frac{\partial}{\partial x} \left( -\mu(s) \frac{\partial b}{\partial x} + b\chi(s) \frac{\partial s}{\partial x} \right)$$

The nutrient diffuses passively and is eventually eaten by the bacterial. We can also write a balance law for the nutrient, therefore:

(13)

$$\frac{\partial s}{\partial t} = D \frac{\partial^2 s}{\partial x^2} - k(s)b$$

where  $D$  is the molecular diffusivity of the food and  $k(s)$  is the rate of nutrient consumption per bacterium. Definitely, in general we expect  $k(s)$  to fall to zero as  $s$  falls to 0, because it is clear that no consumption can be possible when no food is present.

In this paper only the case when the diffusive flux of nutrient is negligible will be taken into consideration, for the sake of simplicity and because of practical reasons, namely because both the random motility and the chemotactic sensitivity are so much larger than the nutrient diffusivity,  $D$ ; this implies that we should regard the nutrient as motionless while the bacteria sweep through it.

### Seeking for travelling wave solutions

We are going to seek for a travelling wave solution of (12) and (13). We first assume that

$b(x, t) = B(z)$  and  $s(x, t) = S(z)$ , where  $z = x - ct$ . By substituting into the named equations we obtain:

(14)

$$-c \frac{dB}{dZ} = -\frac{d}{dZ} \left( -\mu \frac{dB}{dZ} + b\chi \frac{dS}{dZ} \right)$$

and

(15)

$$-c \frac{dS}{dZ} = -kB$$

As it would be rather difficult to treat (14) in the form above, we can integrate it once. Evaluating at  $z$  tending to infinity, we see that the constant of the integration must vanish, therefore we have:

$$(16) \quad -cB = \mu \frac{dB}{dZ} - b\chi \frac{dS}{dZ}$$

Further on, we can eliminate  $dS/dz$  in (16), using (15). This is how we obtain a second order, nonlinear, autonomous ODE system:

$$(17) \quad \frac{dS}{dZ} = \frac{k(s)}{c} B$$

$$(18) \quad \frac{dB}{dZ} = \frac{B}{c} \mu(s)(k(s)\chi(s)B - c^2)$$

Now, we are interested in solutions  $(S(z), B(z))$  for the system above that are non-negative and bounded for all values of the wave variable  $z$ . Actually, since we are interested in isolated propagating swarms of bacteria, we are interested in solutions for which  $B(z)$  tends to 0, both as  $z$  tends to  $\infty$  and as  $z$  tends to  $-\infty$ .

Taking into consideration the first of the equations of the system determined above (17, 18), we notice that as  $z$  tends to  $-\infty$ ,  $S(z)$  must decrease monotonically. But  $S(z)$  cannot fall below zero, because this way we would lose the biological interpretation of it. Thus, the only possibility left is

$$(19) \quad \lim_{z \rightarrow -\infty} S(z) = s_c = \text{const}$$

The biological interpretation will also help us, this time, in simplifying the mathematical assumptions. As we do not know what the unconsumed nutrient concentration left in the wake of a band might be, but we know that it is very small, we can lose no generality by assuming  $s_c=0$ , because, if we have  $s_c>0$  we can any time introduce a new dependent variable  $s'=s- s_c$ , and thus new coefficient functions

$$(20) \quad k'(s') = k(s' + s_c), \quad \mu'(s') = \mu(s' + s_c), \quad \chi'(s') = \chi(s' + s_c)$$

Thus, the transformed version of equations (17, 18) will look similar, but with all the coefficients being primed. We will have then

$$(21) \quad \lim_{z \rightarrow -\infty} s'(z) = 0,$$

We wish to discover what phenomenological coefficient functions

$$(22) \quad k(s), \chi(s), \mu(s)$$

make such solutions possible. Keller and Odell considered a class of representative functions, namely

$$(23) \quad k(s) = k_0 s^\alpha, \quad \mu(s) = \mu_0 s^r, \quad \chi(s) = \frac{\delta_0}{s^p},$$

where  $k_0$ ,  $\mu_0$  and  $\delta_0$  are positive constants, and  $\alpha$ ,  $r$  and  $p$  are constants. The biological meanings of these parameters can be briefly synthesised in:  $k_0$ =the

consumption rate constant,  $\mu_0$  = random motility coefficient and  $\delta_0$  = the ambient nutrient concentration. One of the reasons for working only with this special class of functions is that the actual analysis becomes much easier.

Our system (17, 18) becomes in this way:

$$(25) \quad \frac{ds}{dz} = \frac{k_0}{c} s^\alpha B$$

$$(26) \quad \frac{dB}{dZ} = \frac{1}{c} \mu_0 s^{-r} (k_0 \delta_0 S^{\alpha-p} B - c^2) B$$

Next, we will discuss the necessary condition on the nutrient consumption rate,

$$(27) \quad k(s) = k_0 s^\alpha$$

Now, if we let A be the cross-sectional area of the capillary tube and consequently we denote by N the number of bacteria in the band, we arrive at:

$$(28) \quad N = A \int_{-\infty}^{\infty} B(z) dz$$

Using equation (25) and (28) we derive the following;

$$(29) \quad N = A \frac{c}{k_0} \int_{z=-\infty}^{z=\infty} \int \left( \frac{dS}{dz} \right) dz = \frac{cA}{k_0} \int_{S=0}^{S=S(\infty)=S_0} S^{-\alpha} dS = \frac{c}{k_0(1-\alpha)} S^{1-\alpha} \Big|_{S=0}^{S=S_0}$$

From this last expression we see that the band can have a finite number of bacteria in it only if (from  $S^{1-\alpha}$ )

$$(30) \quad \alpha < 1$$

We will keep in mind this first solution for the moment.

In this case we have

$$(31) \quad c = \frac{k_0(1-\alpha)N}{S_0^{1-\alpha}}$$

Practically this last equation represents the fact that the band speed increases as the number of bacteria in the band increases or as the rate at which they eat increases, or as the ambient food concentration decreases ( $S_0$ ).

This equation is also important for the fact that if  $k_0$ ,  $N$  and  $S_0$  are fixed, then  $c$  is fixed also regardless of the value of  $\delta$ .

What is very interesting to do now is finding the steady states of system (25, 26).

### Steady states, nullclines, phase portrait analysis

In general, the easiest and indicated way of seeking and interpreting solutions to ODE systems is the phase plane analysis. Thus, also in the present paper, with reference to the particular ODE consisting of equations (25, 26), phase plane portraits have been drawn with the aid of the newphase program developed by professors Richard Mansfield and Frits Beukers.

It is known and proven that, in general (any biological phenomena studied in order to stress the propagating effects), biologically meaningful propagating solutions are only obtained if the phase plane portrait corresponding to travelling waves admits a bounded trajectory that is contained entirely in the positive population quadrant.

Now, of course, we are not including among these orbits the trivial bounded trajectory: the point  $(0, 0)$ . In our particular case, after analysing the steady states we will be able to eliminate from the first step those cases that are not bounded and contained entirely in the first quadrant. Sometimes applied mathematics is much easier than pure theoretical mathematics...

Let us write again the system that we are working with:

$$(25) \quad \frac{ds}{dz} = \frac{k_0}{c} s^\alpha B$$

$$(26) \quad \frac{dB}{dZ} = \frac{1}{c} \mu_0 s^{-r} (k_0 \delta_0 S^{\alpha-p} B - c^2) B$$

$(S, B) = (S_0, 0)$  is a steady point for this system for all positive values of  $S_0$ . In other words, every single point on the positive  $S$ -axis is a steady state point, and, moreover, there are no other steady state points.

The next step is to find the  $B$ -nullcline. This nullcline is given by the curve:

$$(32) \quad k_0 \delta_0 S^{\alpha-p} B - c^2 = 0$$

which can be also written consequently as follows:

$$(33) \quad B = \frac{c^2}{k_0 \delta_0} S^{p-\alpha}$$

Now if we divide the second equation of the system by the first, we obtain an equation giving the slope of the trajectories at the point  $(S, B)$ .

$$(34) \quad \frac{dB}{dS} = \frac{1}{k_0 \mu_0} S^{-(r+\alpha)} (k_0 \delta_0 S^{\alpha-p} B - c^2)$$

From this last equation we see that, on the S-axis ( $B=0$ ), all the trajectories have the negative slope:

$$(35) \quad \left. \frac{dB}{dS} \right|_{B=0} = \frac{-c^2}{k_0 \mu_0 S^{r+\alpha}}$$

The next step is to build phase portraits for our system (25, 26) for different choices of the coefficients. The purpose is definitely to see which of these choices permit bounded positive solutions such that

$$(36) \quad \lim_{|z| \rightarrow \infty} B(z) = 0$$

(now we have defined exactly our target)

However, our job is made easier because, indifferent of the case discussed, there will be a trajectory that approach steady state points (these being any points on the S-axis) as  $z \rightarrow \infty$ . Therefore we seek such solutions for which  $B(z) > 0$  as  $z \rightarrow -\infty$ . The papers containing the phase plane portraits are attached at the end of the document.

### Discussing the different cases

#### Case I, $p-\alpha < 0$

We can see from the phase plane analysis as well as from very simple observations that the B-nullcline tends to  $\infty$  as S tends to  $0^+$ . Therefore, just using (34) above and the location of the B nullcline, we can see (without bothering any longer with any detailed analysis of where the trajectories go as z tends to  $-\infty$ ), that B(z) cannot possibly become zero when z tends to 0. Therefore we can already reject this case.

#### Case II, $p-\alpha = 0$

In this case the B-nullcline will be horizontal and will intersect the B-axis at a finite "height", nonzero however. This can be seen very well from the phase plane portrait attached as appendix to this paper. If we pick any trajectory approaching a steady state point on the S-axis, as z tends to, this trajectory can never approach  $B=0$  as it is followed backward. Therefore this case is also rejected without bothering with any more detailed analysis of the trajectories as z tends to  $-\infty$ .

#### Case III, $p-\alpha > 0$

This case is the most complex one; first of all it can be subdivided into other 3 subclasses:  $0 < p-\alpha < 1$ ,  $p-\alpha = 1$  and  $p-\alpha > 1$ . In each case the B nullcline intersects the origin (0,0) and rises as S increases. All these cases can be easily visualised on the phase portraits attached as appendix.

While in the second subclass we have the B nullcline is a straight line, for the other subclasses the B-nullcline will look like a concave curve, downward sloping (in subclass I), respectively upward sloping (in subclass III).

Now, whatever trajectory we pick in the first quadrant, and we trace it backwards (so z decreasing), B decreases, therefore there is a chance that, as z tends



to  $-\infty$ , the point (0,0) is approached. So, we have found one condition as necessary for the travelling bands:  $p > \alpha$

We also had a previous condition stating that we can have a finite number of bacteria in the band only if  $\alpha < 1$ .

Having the two previous conditions accomplished, namely  $p > \alpha$  and  $\alpha < 1$ , we claim that  $r + p > 1$  is necessary and sufficient so that  $S \rightarrow 0$  and  $B \rightarrow 0$ , as  $z \rightarrow -\infty$ . In other words, these 3 conditions put together are sufficient to guarantee the existence of a well-behaved travelling band containing finitely many bacteria.

Segel gives the sufficient condition as an example. Practically, the steps taken to prove it are the following:

The assumption is proven by the method of reduction to absurd. We have as hypothesis the 3 conditions reminded above. We assume that B does not approach 0 as  $S \rightarrow 0$ . Then, as B stays positive, in equation (34) the  $S^{\alpha-p} B$  term overwhelms the  $c^2$ , as  $S \rightarrow 0$  (this is because  $S^{\alpha-p} \rightarrow +\infty$  as  $S \rightarrow 0$ ). Then, near  $S \rightarrow 0$ , equation (34) becomes

$$\frac{dB}{dS} \cong \frac{\delta_0}{\mu_0} S^{-r-\alpha+p} B, \text{ which can be written as } \int dB/B \cong \int (\delta_0 / \mu_0) S^{-r-p} dS.$$

Next, by integrating our last equation, we get:

$$\ln B \cong \frac{(\delta_0 / \mu_0)}{1-r-p} S^{1-r-p} + const$$

By writing this equation in the exponential form, we get the following:

$$B = Qe^{\frac{\delta_0 / \mu_0}{1-r-p} S^{1-r-p}}$$

Next, knowing that  $r + p > 1$  (from our hypothesis) we get  $\frac{\delta_0 / \mu_0}{1-r-p} < 0$ . Thus, the

exponent in the equation above tends to  $-\infty$ . Since  $e^{-\infty} = 0$ , we have  $B \rightarrow 0$  as  $S \rightarrow 0$ , contradicting with what we assumed about B. This contradiction proves the sufficiency condition.

Unfortunately I did not manage to prove the necessary condition. I took it therefore as true. Practically the idea is to use the same method of reduction to absurd and arrive at a contradiction.

We found that the ODE system (25, 26) has a positive bounded solution B (z) such that  $\lim B(z) = 0$  as  $|z| \rightarrow \infty$  and thus we have a solitary pulse travelling band solution

(with  $\int_{-\infty}^{\infty} b(x,t) dz$  finite) if and only if  **$\alpha < 1$ ,  $p > \alpha$ , and  $p + r > 1$** .

### Interpretation of the results

We can biologically interpret these results. The condition  $\alpha < 1$  means, quoting Keller and Odell, that “however appealing the  $\alpha = 1$  choice may be, making  $k(s) = k_0 s$  near  $s = 0$ , it will not permit exact travelling band solutions”. It is however possible to get travelling bands if  $k(s)$  vanishes abruptly (thus with infinite slope) at  $s = 0$ . For example for  $\alpha = .5$ , so  $k(s) = k_0 \sqrt{s}$ .

Next,  $k(s)$  must vanish as  $s \rightarrow 0$  (because no food implies no consumption); thus for keeping the biological reality. Therefore, we need  $0 < \alpha < 1$ , and thus  $p > \alpha > 0$ .

This means that for exact travelling band solutions, the chemotactic "sensitivity",  $\chi(s) = \delta_0 / s^p$ , must become infinite as  $s \rightarrow 0$ .

For the third condition the interpretation given by Segel is the following: the larger  $r$ , the faster the random motility coefficient  $\mu(s) = \mu_0 s^r$  vanishes as  $s \rightarrow 0$  at the trailing edge. Furthermore, the, the larger  $r$ , the smaller has  $p$  to be. Since  $\chi(s)$  needs to be large as  $s \rightarrow 0$ , precisely to balance random motility, the trade-off suggested by  $p+r > 1$  seems to make sense.

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